Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis I

Final Examination | Date : Nov. 7, 2016
Maximum marks: 100 | Time: 3 hours

Here the set of natural numbers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}$ and the set of real numbers is denoted by $\mathbb{R}$.

1. Let K be the set of functions from $\{0, 1\}$ to $\mathbb{N}$. Show that K is countable. Let M be the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Show that M is uncountable. [15]

2. Let $\{v_n\}_{n\geq 1}$ be the sequence defined by $v_1 = 1$ and $v_{n+1} = \sqrt{v_n^2 + \frac{1}{2^n}}$ for $n \geq 1$. Show that $\lim_{n\to\infty} v_n$ exists. Find the limit. [15]

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \min\{f(x), g(x)\}$ is a continuous function. Show that the converse is not true. Show that if f, g are differentiable at c and $f(c) \neq g(c)$ then h is differentiable at c. [15]

4. Let $m : (0, 1) \to \mathbb{R}$ be a continuous function. Suppose $\{m(x) : x \in \mathbb{R}\} \subseteq \mathbb{N}$. Show that m is a constant function. [15]

5. Suppose $k \in \mathbb{N}$ and $B_1, B_2, \ldots, B_k$ are strictly positive real numbers. Show that

   (i) $\lim_{n\to\infty} k^{\frac{1}{n}} = 1$;

   (ii) $\lim_{n\to\infty}(B_1^n + B_2^n + \cdots + B_k^n)^{\frac{1}{n}} = B$ where $B = \max\,\{B_j : 1 \leq j \leq k\}$. [15]

6. Let a, b be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for every $x \in (a, b)$. (i) Show that f is one to one; (ii) Show that either $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. [15]

7. State and prove Taylor's theorem for real valued functions on open subintervals of $\mathbb{R}$. [15]